

# Nonarchimedean Green Functions and Dynamics on Projective Space

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ABSTRACT. Let  $\varphi : \mathbb{P}_K^N \rightarrow \mathbb{P}_K^N$  be a morphism of degree  $d \geq 2$  defined over a field  $K$  that is algebraically closed field and complete with respect to a nonarchimedean absolute value. We prove that a modified Green function  $\hat{g}_\varphi$  associated to  $\varphi$  is Hölder continuous on  $\mathbb{P}^N(K)$  and that the Fatou set  $\mathcal{F}(\varphi)$  of  $\varphi$  is equal to the set of points at which  $\hat{g}_\varphi$  is locally constant. Further,  $\hat{g}_\varphi$  vanishes precisely on the set of points  $P$  such that  $\varphi$  has good reduction at every point in the forward orbit  $\mathcal{O}_\varphi(P)$  of  $P$ . We also prove that the iterates of  $\varphi$  are locally uniformly Lipschitz on  $\mathcal{F}(\varphi)$ .

## INTRODUCTION

Let  $K$  be an algebraically closed field that is complete with respect to a nontrivial nonarchimedean absolute value  $|\cdot|$ . An example of such a field is  $\mathbb{C}_p$ , the completion of the algebraic closure of  $\mathbb{Q}_p$ .

Let  $\varphi : \mathbb{P}_K^1 \rightarrow \mathbb{P}_K^1$  be a rational function of degree  $d \geq 2$  defined over  $K$ . The absolute value on  $K$  induces a natural metric on  $\mathbb{P}^1(K)$ , and nonarchimedean dynamics is the study of the iterated action of  $\varphi$  on  $\mathbb{P}^1(K)$  relative to this metric. The family of iterates  $\{\varphi^n\}_{n \geq 0}$  divides  $\mathbb{P}^1(K)$  into two disjoint (possibly empty) subsets, the Fatou set  $\mathcal{F}(\varphi)$  and the Julia set  $\mathcal{J}(\varphi)$ . The Fatou set is the the largest open subset of  $\mathbb{P}^1(K)$  on which the family is equicontinuous, and the Julia set is the complement of the Fatou set. There has been considerable interest in nonarchimedean dynamics on  $\mathbb{P}^1$  in recent years, see for example [1, 3, 4, 5, 6, 7, 12, 13, 17, 18, 19, 20].

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In this article we investigate aspects of nonarchimedean dynamics on higher dimensional projective spaces. For points

$$P = (x_0 : \cdots : x_N) \in \mathbb{P}^N(K) \quad \text{and} \quad Q = (y_0 : \cdots : y_N) \in \mathbb{P}^N(K)$$

we define the *chordal distance* from  $P$  to  $Q$  to be

$$\Delta(P, Q) = \frac{\max_{0 \leq i, j \leq N} |x_i y_j - x_j y_i|}{\max\{|x_0|, \dots, |x_N|\} \max\{|y_0|, \dots, |y_N|\}}.$$

This defines a nonarchimedean metric on  $\mathbb{P}^N(K)$ . As in the one dimensional case, for any  $K$ -morphism  $\varphi : \mathbb{P}^N \rightarrow \mathbb{P}^N$  of degree  $d \geq 2$  we define the Fatou set  $\mathcal{F}(\varphi)$  to be the largest open set on which the iterates of  $\varphi$  are equicontinuous, and the Julia set  $\mathcal{J}(\varphi)$  is the complement of the Fatou set. (See Section 7 for the precise definitions.) Also for convenience, for any vector  $x = (x_0, \dots, x_N) \in K^{N+1}$ , we write  $\|x\| = \max |x_i|$  for the sup norm.

Over the complex numbers, pluri-potential theory has played a key role in the study of complex dynamics on  $\mathbb{P}^N(\mathbb{C})$ . One of the primary goals of this paper is to develop an analogous theory in the nonarchimedean setting. For a given morphism  $\varphi : \mathbb{P}_K^N \rightarrow \mathbb{P}_K^N$  of degree  $d \geq 2$ , let

$$\Phi : K^{N+1} \longrightarrow K^{N+1}$$

be a lift of  $\varphi$ . Then as in the complex case (cf. [21]) one defines the *Green function* (or *potential function*) associated to  $\Phi$  by the limit

$$G_\Phi(x) = \lim_{n \rightarrow \infty} \frac{1}{d^n} \log \|\Phi^n(x)\|. \quad (1)$$

The existence of the limit and the relation of  $G_\Phi$  to canonical local height functions is explained in [15]. We also define a *modified Green function*

$$\hat{g}_\Phi : \mathbb{P}^N(K) \rightarrow \mathbb{R}, \quad \hat{g}_\Phi(P) = G_\Phi(x) - \log \|x\|, \quad (2)$$

that is well-defined independent of the choice of the lift  $x \in K^{N+1}$  of  $P \in \mathbb{P}^N(K)$ . The main results of this paper are summarized in the following theorem.

**Theorem 1.** *Let  $\varphi : \mathbb{P}^N \rightarrow \mathbb{P}^N$  be a morphism of degree  $d \geq 2$  as above and let  $\hat{g}_\Phi$  be an associated Green function on  $\mathbb{P}^N(K)$  as defined by (1) and (2).*

- (a) *The function  $\hat{g}_\Phi$  is Hölder continuous on  $\mathbb{P}^N(K)$ .*
- (b) *The Fatou set of  $\varphi$  is characterized by*

$$\mathcal{F}(\varphi) = \{P \in \mathbb{P}^N(K) : \hat{g}_\Phi \text{ is locally constant at } P\}.$$

- (c) *The Fatou set of  $\varphi$  is equal to the set of points  $P$  such that the iterates of  $\varphi$  are locally uniformly Lipschitz at  $P$ , i.e., such that there is a neighborhood  $U$  of  $P$  and a constant  $C$  so that*
- $$\Delta(\varphi^n(Q), \varphi^n(R)) \leq C\Delta(Q, R) \quad \text{for all } Q, R \in U \text{ and all } n \geq 0.$$
- (d)  *$\hat{g}_\Phi(P) = 0$  if and only if  $\varphi$  has good reduction at every point in the forward orbit  $\mathcal{O}_\varphi(P)$ . Further, the set of such points is an open set and is contained in the Fatou set  $\mathcal{F}(\varphi)$ .*

As an immediate corollary of Theorem 1(b) and the fact (Corollary 21) that  $\varphi$  is an open mapping in the nonarchimedean topology, we obtain the invariance of the Fatou and Julia sets.

**Corollary 2.** *The Fatou set  $\mathcal{F}(\varphi)$  and the Julia set  $\mathcal{J}(\varphi)$  are forward and backward invariant for  $\varphi$ .*

*Remark 3.* Parts (a) and (b) of Theorem 1 are analogous to results in pluri-potential theory over  $\mathbb{C}$ . Thus if  $\varphi : \mathbb{P}_{\mathbb{C}}^N \rightarrow \mathbb{P}_{\mathbb{C}}^N$  is a morphism of degree  $d \geq 2$  and  $\Phi : \mathbb{C}^{N+1} \rightarrow \mathbb{C}^{N+1}$  is a lift of  $\varphi$ , the classical Green function  $G_\Phi : \mathbb{C}^{N+1} \rightarrow \mathbb{R}$  associated to  $\Phi$  is defined by the same limit (1) that we are using in the nonarchimedean setting. It is then well known that  $G_\Phi$  is Hölder continuous on  $(\mathbb{C}^{N+1})^*$  and that the Fatou set of  $\varphi$  is the image in  $\mathbb{P}^N(\mathbb{C})$  of the set

$$\{x \in (\mathbb{C}^{N+1})^* : G_\Phi \text{ is pluri-harmonic at } x\}.$$

See for example [21].

We note that applying  $dd^c$  to  $G_\Phi$  gives the *Green current*  $T_\Phi$  on  $\mathbb{P}^N(\mathbb{C})$  and that the invariant measure associated to  $\varphi$  is obtained as an intersection of  $T_\Phi$ . The invariant measure is of fundamental importance in studying the complex dynamics of  $\varphi$ . An analogous theory has been developed on  $\mathbb{P}^1$  in the nonarchimedean setting (see for example [9, 2, 11, 23]) and it would be interesting to extend this to  $\mathbb{P}^N$ .

Finally, we mention that the Hölder continuity of  $G_\Phi$  over  $\mathbb{C}$  is used to estimate the Hausdorff dimension of the Julia set.

The proof of Theorem 1 is given in Theorem 18, Theorem 24, and Proposition 32. The proofs of (a) and (b) follow the complex proofs to some extent, but there are also parts of the proofs that are specifically nonarchimedean, especially where compactness arguments over  $\mathbb{C}$  are not applicable to nonlocally compact fields such as  $\mathbb{C}_p$ . Further, we are able to make most constants in this article explicit in terms of the Macaulay resultant of  $\Phi$ . (See Section 2 for the definition and basic properties of the Macaulay resultant.)

The organization of this paper is as follows. In Section 1 we define the chordal metric on  $\mathbb{P}^N(K)$  and prove some of its properties. In

Section 2 we consider Lipschitz continuity and show in particular that  $\varphi : \mathbb{P}^N \rightarrow \mathbb{P}^N$  is Lipschitz continuous with an explicit Lipschitz constant. In Section 3 we review the definition and basic properties of the Green function  $G_\Phi$  and use them to deduce various elementary properties of the modified Green function  $\hat{g}_\Phi$ . In Section 4 we show that  $\hat{g}_\Phi$  is Hölder continuous with explicit constants. In Section 5 we prove that morphisms are open mappings in the nonarchimedean setting. In Section 6 we recall some facts from nonarchimedean analysis. In Section 7 we define the Fatou and Julia sets in terms of equicontinuity for the family  $\{\varphi^n\}$  with respect to the chordal metric. In Section 8 we characterize the Fatou set in terms of the Green function and give some applications, including the backward and forward invariance of  $\mathcal{F}(\varphi)$  and  $\mathcal{J}(\varphi)$ . Finally in Section 9 we relate the Fatou set and the vanishing of  $\hat{g}_\Phi$  to sets of points at which  $\varphi$  has good reduction.

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## 1. THE CHORDAL METRIC ON $\mathbb{P}^N$

For the remainder of this paper we fix an algebraically closed field  $K$  that is complete with respect to a nontrivial nonarchimedean absolute value  $|\cdot|$ . We extend the absolute value on  $K$  to the sup norm on  $K^{N+1}$ , which we denote by

$$\|x\| = \max\{|x_0|, \dots, |x_N|\} \quad \text{for } x = (x_0, \dots, x_N) \in K^{N+1}.$$

We also write

$$\pi : (K^{N+1})^* \rightarrow \mathbb{P}^N(K)$$

for the natural projection map.

**Definition.** Let  $P, Q \in \mathbb{P}^N(K)$  and choose lifts  $x, y \in (K^{N+1})^*$  for  $P$  and  $Q$ , i.e.,  $\pi(x) = P$  and  $\pi(y) = Q$ . The (*nonarchimedean*) *chordal distance from  $P$  to  $Q$*  is defined by

$$\Delta(P, Q) = \frac{\max_{0 \leq i, j \leq N} |x_i y_j - x_j y_i|}{\|x\| \cdot \|y\|}.$$

By homogeneity, it is clear that  $\Delta(P, Q)$  is independent of the choice of lifts for  $P$  and  $Q$ .

*Remark 4.* The chordal distance is an example of a  $v$ -adic (arithmetic) distance function as defined in [22, §3], although we note that the function  $\delta$  defined in [22] is logarithmic, i.e.,  $\delta(P, Q) = -\log \Delta(P, Q)$ . Further, all of the distance and height functions in [22] are Weil functions

in the sense that they are only defined up to addition of a bounded function that depends on the underlying variety. So to be precise, the logarithmic chordal distance  $-\log \Delta$  is a particular function in the equivalence class of arithmetic distance functions  $\delta$  on  $\mathbb{P}^N$ .

**Lemma 5.** *The chordal distance  $\Delta$  defines a nonarchimedean metric on  $\mathbb{P}^N(K)$ . Further, it is bounded by  $\Delta(P, Q) \leq 1$ .*

*Proof.* It is immediate from the definition that  $\Delta(P, Q) \geq 0$  and that it is equal to 0 if and only if  $P = Q$ . Further,

$$\max_{0 \leq i, j \leq N} |x_i y_j - x_j y_i| \leq \max_{0 \leq i, j \leq N} \max\{|x_i y_j|, |x_j y_i|\} \leq \|x\| \cdot \|y\|,$$

which proves that  $\Delta(P, Q) \leq 1$ . It remains to verify that  $\Delta$  satisfies the strong triangle inequality.

Let  $R \in \mathbb{P}^N(K)$  be a third point and lift it to  $z \in (K^{N+1})^*$ . Multiplying each lift by an appropriate element of  $K^*$ , we may normalize the lifts to satisfy

$$\|x\| = \|y\| = \|z\| = 1.$$

Consider the identity

$$\begin{aligned} (x_i z_k - x_k z_i) y_j \\ = (x_i y_j - x_j y_i) z_k + (y_i z_k - y_k z_i) x_j + (x_j y_k - x_k y_j) z_i. \end{aligned} \quad (3)$$

Since  $\|y\| = 1$ , there is a  $j_0$  with  $|y_{j_0}| = 1$ . Then (3) with  $j = j_0$  gives

$$|x_i z_k - x_k z_i| \leq \max\{\Delta(P, Q), \Delta(Q, R)\}.$$

Taking the maximum over  $i$  and  $k$  yields the strong triangle inequality,

$$\Delta(P, R) \leq \max\{\Delta(P, Q), \Delta(Q, R)\}. \quad \square$$

In the remainder of this section we develop some basic properties of the chordal metric on  $\mathbb{P}^N(K)$ . We begin with some notation that will be used throughout the remainder of this paper.

Let  $M \geq 1$  be an integer, typically equal to either  $N$  or  $N + 1$ . For  $a \in K^M$  and  $r > 0$ , the *open polydisk* and the *closed polydisk* centered at  $a$  with radius  $r$  are defined, respectively, by

$$\begin{aligned} B(a, r) &= \{x \in K^M : \|x - a\| < r\}, \\ \bar{B}(a, r) &= \{x \in K^M : \|x - a\| \leq r\}. \end{aligned}$$

Similarly, for  $P \in \mathbb{P}^N(K)$  and  $1 \geq r > 0$ , we define the *open disk* and the *closed disk* centered at  $P$  with radius  $r$  to be, respectively,

$$D_r(P) = \{Q \in \mathbb{P}^N(K) : \Delta(P, Q) < r\},$$

$$\bar{D}_r(P) = \{Q \in \mathbb{P}^N(K) : \Delta(P, Q) \leq r\}.$$

Despite the terminology, all four of the sets  $B(a, r)$ ,  $\bar{B}(a, r)$ ,  $D_r(P)$ , and  $\bar{D}_r(P)$  are both open and closed in the topology induced by  $\|\cdot\|$  on  $K^M$  and by the chordal metric  $\Delta$  on  $\mathbb{P}^N(K)$ . We also embed  $K^N$  into  $\mathbb{P}^N(K)$  via the map

$$\sigma : K^N \hookrightarrow \mathbb{P}^N(K), \quad (x_1, \dots, x_N) \mapsto (1 : x_1 : \dots : x_N).$$

**Lemma 6.** *Let  $P, Q \in \mathbb{P}^N(K)$  be points satisfying  $\Delta(P, Q) < 1$ . Choose a lift  $x \in (K^{N+1})^*$  for  $P$  and a lift  $y \in (K^{N+1})^*$  for  $Q$ . and let  $0 \leq k \leq N$  be an index. Then*

$$|x_k| = \|x\| \quad \text{if and only if} \quad |y_k| = \|y\|.$$

*Proof.* We may assume that  $\|x\| = \|y\| = 1$ . Assume that  $|x_k| = 1$  and choose an index  $j$  such that  $|y_j| = 1$ . Then

$$|x_k y_j - x_j y_k| \leq \Delta(P, Q) < 1 \quad \text{and} \quad |x_k y_j| = 1,$$

so the strong triangle inequality implies that  $|x_j y_k| = 1$ . But  $|x_j| \leq 1$  and  $|y_k| \leq 1$ , so we must have  $|y_k| = 1$ .  $\square$

The next lemma shows that the usual metric  $\|\cdot\|$  and the chordal metric  $\Delta$  are the same on the closed unit polydisk  $\bar{B}(0, 1)$  in  $K^N$ .

**Lemma 7.** (a) *The restriction of  $\sigma$  to  $\bar{B}(0, 1)$  is an isometry,*

$$\Delta(\sigma(x), \sigma(y)) = \|x - y\| \quad \text{for all } x, y \in \bar{B}(0, 1).$$

(b) *Let  $x \in \bar{B}(0, 1)$  and  $1 > r > 0$ . Then the maps*

$$\sigma : \bar{B}(x, r) \rightarrow \bar{D}_r(\sigma(x)) \quad \text{and} \quad \sigma : B(x, r) \rightarrow D_r(\sigma(x))$$

*are isometric isomorphisms.*

*Proof.* Let  $x, y \in \bar{B}(0, 1)$ . Then  $\|\sigma(x)\| = \|\sigma(y)\| = 1$ , so

$$\Delta(\sigma(x), \sigma(y)) = \max_{0 \leq i, j \leq N} \{|x_i y_j - x_j y_i|\},$$

where for convenience we set  $x_0 = y_0 = 1$ . In particular, putting  $j = 0$  gives

$$\Delta(\sigma(x), \sigma(y)) \geq \max_{0 \leq i \leq N} \{|x_i - y_i|\} = \|x - y\|.$$

Further, we note that

$$|x_i y_j - x_j y_i| = |x_i(y_j - x_j) + x_j(x_i - y_i)| \leq \max\{|y_j - x_j|, |x_i - y_i|\}.$$

Taking the maximum over all  $i$  and  $j$  gives

$$\Delta(\sigma(x), \sigma(y)) \leq \max_{0 \leq i \leq N} |x_i - y_i| = \|x - y\|, .$$

which gives the opposite inequality and completes the proof of (a).

By assumption  $x \in \bar{B}(0, 1)$  and  $r < 1$ , so the triangle inequality implies that  $\bar{B}(x, r) \subset \bar{B}(0, 1)$ . Then (a) tells us that  $\sigma$  is an isometry on  $\bar{B}(x, r)$ , so in particular  $\sigma$  maps  $\bar{B}(x, r)$  injectively and isometrically into  $\bar{D}_r(\sigma(x))$ .

It remains to check that the map is surjective. Let  $Q \in \bar{D}_r(\sigma(x))$  and lift  $Q$  to  $b = (b_0, b_1, \dots, b_N)$ . We know that  $\|\sigma(x)\| = 1$  and that the first coordinate of  $\sigma(x)$  equals 1, and also  $\Delta(Q, \sigma(x)) \leq r < 1$ , so Lemma 6 tells us the  $|b_0| = \|b\|$ . Then the point

$$y = \left( \frac{b_1}{b_0}, \frac{b_2}{b_0}, \dots, \frac{b_N}{b_0} \right) \quad \text{is in } \bar{B}(0, 1) \text{ and satisfies } \sigma(y) = Q.$$

Finally, since  $x, y \in \bar{B}(0, 1)$ , we can use (a) again to compute

$$\|x - y\| = \Delta(\sigma(x), \sigma(y)) = \Delta(\sigma(x), Q) \leq r,$$

so in fact  $y \in \bar{B}(x, r)$ . This proves that  $\sigma(\bar{B}(x, r)) = \bar{D}_r(\sigma(x))$ , which completes the first part of (b). The second part is proven similarly.  $\square$

**Proposition 8.**  $\mathbb{P}^N(K)$  is complete with respect to the chordal metric  $\Delta$ . (As always, we are assuming that the field  $K$  is complete.)

*Proof.* Fix some  $r < 1$ , say  $r = \frac{1}{2}$ . Let  $(P_i)_{i \geq 1}$  be a Cauchy sequence in  $\mathbb{P}^N(K)$  and fix an  $n$  so that  $\Delta(P_i, P_j) \leq r$  for all  $i, j \geq n$ . In particular, the truncated sequence  $(P_i)_{i \geq n}$  lies in the disk  $\bar{D}_r(P_n)$ . Re-ordering the coordinates if necessary, we can assume that there is a lift  $x \in \bar{B}(0, 1)$  of  $P_n$ . Then Lemma 7(b) tells us that  $\bar{D}_r(P_n)$  is isometrically isomorphic to  $B(x, r)$ . But  $B(x, r) \subset K^N$  and  $K^N$  is complete, hence  $\bar{D}_r(P_n)$  is also complete.  $\square$

## 2. LIPSCHITZ CONTINUITY OF MORPHISMS

In this and subsequent sections, we say that an element  $a \in K$  is  $K$ -integral if  $|a| \leq 1$  and we say that  $a$  is a  $K$ -unit if  $|a| = 1$ .

Associated to any collection of homogeneous polynomials

$$\Phi = (\Phi_0, \dots, \Phi_N) : \mathbb{A}^{N+1} \longrightarrow \mathbb{A}^{N+1}$$

in  $N + 1$  variables is a polynomial  $\text{Res}(\Phi)$  (with integer coefficients) in the coefficients of  $\Phi_0, \dots, \Phi_N$  whose vanishing is equivalent to the collection  $\Phi_0, \dots, \Phi_N$  having a nontrivial common zero. See [15, §1.1] for a summary of the basic properties of this *Macaulay resultant*  $\text{Res}(\Phi)$  and [14] for full details and proofs. We recall the following useful result.

**Proposition 9.** *Let  $\Phi_0, \dots, \Phi_N \in K[X_0, \dots, X_N]$  be a collection of homogeneous polynomials with  $K$ -integral coefficients. Then*

$$|\operatorname{Res}(\Phi)| \cdot \|x\|^d \leq \|\Phi(x)\| \leq \|x\|^d \quad \text{for all } x \in \mathbb{A}^{N+1}(K).$$

*Proof.* See [15, Proposition 6(b)].  $\square$

**Definition.** Let  $\varphi : \mathbb{P}_K^N \rightarrow \mathbb{P}_K^N$  be a morphism defined over  $K$  and let  $\Phi : \mathbb{A}_K^{N+1} \rightarrow \mathbb{A}_K^{N+1}$  be a lift of  $\varphi$ . We say that  $\Phi$  is a *minimal lift* of  $\varphi$  if all of its coefficients are  $K$ -integral and at least one coefficient is a  $K$ -unit. Any two minimal lifts differ by multiplication by a  $K$ -unit.

We define a *minimal resultant*  $\operatorname{Res}(\varphi)$  of  $\varphi$  to be the resultant of a minimal lift of  $\varphi$ . Note that  $\operatorname{Res}(\varphi)$  is well defined up to multiplication by a power of a  $K$ -unit, so in particular, the absolute value  $|\operatorname{Res}(\varphi)|$  is well defined independent of the chosen minimal lift.

**Definition.** Let  $\Phi = (\Phi_0, \dots, \Phi_N) : K^{N+1} \rightarrow K^{N+1}$  be a lift of  $\varphi : \mathbb{P}_K^N \rightarrow \mathbb{P}_K^N$ . For each  $i = 0, \dots, N$ , we define the *norm* of  $\Phi_i$  to be the maximum of the absolute values of the coefficients of  $\Phi_i$ . In other words, if  $\Phi_i = \sum a_{i,j_0, \dots, j_N} x_0^{j_0} \cdots x_N^{j_N}$ , then

$$\|\Phi_i\| = \sup_{j_0, \dots, j_N \geq 0} |a_{i,j_0, \dots, j_N}|.$$

We define the *norm* of  $\Phi$  by  $\|\Phi\| = \sup_{0 \leq i \leq N} \|\Phi_i\|$ . In particular, the condition  $\|\Phi\| = 1$  is equivalent to  $\Phi$  being a minimal lift of  $\varphi$ .

We now prove that morphisms of  $\mathbb{P}^N$  over nonarchimedean fields are Lipschitz continuous and give an explicit Lipschitz constant.

**Theorem 10.** *Let  $\varphi : \mathbb{P}^N \rightarrow \mathbb{P}^N$  be a morphism of degree  $d \geq 2$  defined over  $K$ . Then  $\varphi$  is Lipschitz continuous with respect to the chordal metric. More precisely,*

$$\Delta(\varphi(P), \varphi(Q)) \leq |\operatorname{Res}(\varphi)|^{-2} \Delta(P, Q) \quad \text{for all } P, Q \in \mathbb{P}^N(K), \quad (4)$$

where  $\operatorname{Res}(\varphi)$  is a minimal resultant of  $\varphi$ .

*Remark 11.* More generally, any morphism  $\varphi : \mathbb{P}^N \rightarrow \mathbb{P}^M$  is Lipschitz continuous, although the Lipschitz constant depends in a more complicated way on  $\varphi$ .

*Remark 12.* Recall that the map  $\varphi$  has good reduction if its minimal resultant is a  $K$ -unit. (See [15, Section 1.3].) Hence if  $\varphi$  has good reduction, then  $\varphi$  is nonexpanding with respect to the chordal metric, so the Julia set of  $\varphi$  (see Section 7) is empty. This generalizes the well-known result for  $\mathbb{P}^1$ , see for example [16].



*Proof of Theorem 10.* Let  $\Phi = (\Phi_0 : \dots : \Phi_N)$  be a minimal lift of  $\varphi$ . Consider the homogeneous polynomials

$$\Phi_i(X)\Phi_j(Y) - \Phi_j(X)\Phi_i(Y) \in K[X, Y].$$

They are in the ideal generated by

$$\{X_k Y_l - X_l Y_k : 0 \leq k < l \leq N\}.$$

More precisely, there are polynomials  $A_{i,j,k,l}(X, Y)$  whose coefficients are bilinear forms (with integer coefficients) in the coefficients of  $\Phi_i$  and  $\Phi_j$  such that

$$\Phi_i(X)\Phi_j(Y) - \Phi_j(X)\Phi_i(Y) = \sum_{0 \leq k < l \leq N} A_{i,j,k,l}(X, Y)(X_k Y_l - X_l Y_k).$$

Now let  $P, Q \in \mathbb{P}^N(K)$  and write  $P = \pi(x)$  and  $Q = \pi(y)$  as usual with  $\|x\| = 1$  and  $\|y\| = 1$ . Then

$$\begin{aligned} & |\Phi_i(x)\Phi_j(y) - \Phi_j(x)\Phi_i(y)| \\ & \leq \max_{0 \leq k < l \leq N} |A_{i,j,k,l}(x, y)| \cdot |x_k y_l - x_l y_k| \\ & \leq \|\Phi_i\| \cdot \|\Phi_j\| \max_{0 \leq k < l \leq N} |x_k y_l - x_l y_k| \\ & \leq \Delta(P, Q) \quad \text{since } \|\Phi\| = 1 \text{ by assumption.} \end{aligned} \quad (5)$$

Since  $\Phi$  has  $K$ -integral coefficients and  $\|x\| = \|y\| = 1$ , Proposition 9 says that

$$\|\Phi(x)\| \geq |\text{Res}(\Phi)| \quad \text{and} \quad \|\Phi(y)\| \geq |\text{Res}(\Phi)|. \quad (6)$$

Using (5) and (6) in the definition of the chordal distance yields

$$\begin{aligned} \Delta(\varphi(P), \varphi(Q)) &= \frac{\max_{0 \leq i, j \leq N} |\Phi_i(x)\Phi_j(y) - \Phi_j(x)\Phi_i(y)|}{\|\Phi(x)\| \cdot \|\Phi(y)\|} \\ &\leq |\text{Res}(\Phi)|^{-2} \Delta(P, Q). \end{aligned}$$

This completes the proof of Theorem 10.  $\square$

The previous theorem considered the distance from  $\Phi(P)$  to  $\Phi(Q)$ . We next study the variation of the ratio of  $\|\Phi(P)\|$  to  $\|\Phi(Q)\|$ .

**Theorem 13.** *Let  $\varphi : \mathbb{P}^N \rightarrow \mathbb{P}^N$  be a morphism of degree  $d \geq 2$  defined over  $K$ , let  $\Phi : \mathbb{A}_K^{N+1} \rightarrow \mathbb{A}_K^{N+1}$  be a lift of  $\varphi$ , and define a function*

$$g_\Phi : \mathbb{P}^N(K) \longrightarrow \mathbb{R}, \quad g_\Phi(P) = \frac{1}{d} \log \|\Phi(x)\| - \log \|x\| \quad (7)$$

*for any  $x \in \pi^{-1}(P)$ .*

*Then  $g_\Phi$  is Lipschitz continuous with respect to the chordal metric.*

More precisely, for all  $P, Q \in \mathbb{P}^N(K)$  we have

$$|g_\Phi(P) - g_\Phi(Q)| \leq \frac{\log(|\text{Res}(\varphi)|^{-1})}{d|\text{Res}(\varphi)|} \Delta(P, Q). \quad (8)$$

Further,

$$g_\Phi(P) = g_\Phi(Q) \quad \text{if } \Delta(P, Q) < |\text{Res}(\varphi)|. \quad (9)$$

In particular,  $g_\Phi$  is uniformly locally constant. (Note that the norm on the lefthand side of (8) is the usual archimedean absolute value on  $\mathbb{R}$ .)

*Proof.* Homogeneity of  $\Phi$  implies that  $g_\Phi(P)$  is well-defined, independent of the lift of  $P$ . Further, for any constant  $c$  we have

$$g_{c\Phi}(P) = g_\Phi(P) + \frac{1}{d} \log |c|,$$

so the difference  $g_\Phi(P) - g_\Phi(Q)$  is independent of the chosen lift of  $\varphi$ . Hence without loss of generality, we assume that  $\Phi$  is a minimal lift of  $\varphi$ . To ease notation, we let

$$R = |\text{Res}(\varphi)|$$

be the absolute value of the minimal resultant. Note that  $0 < R \leq 1$ .

Let  $P = \pi(x)$  and  $Q = \pi(y)$  with  $\|x\| = \|y\| = 1$  as usual, so in particular Proposition 9 tells us that

$$1 \geq \|\Phi(x)\| \geq R \quad \text{and} \quad 1 \geq \|\Phi(y)\| \geq R. \quad (10)$$

We consider two cases. The first case is for points  $P$  and  $Q$  that are not close together. Suppose that  $\Delta(P, Q) \geq R$ . Then using (10) we find that

$$|g_\Phi(P) - g_\Phi(Q)| = \frac{1}{d} \left| \log \frac{\|\Phi(x)\|}{\|\Phi(y)\|} \right| \leq \frac{1}{d} \log(R^{-1}) \leq \frac{\log(R^{-1})}{dR} \Delta(P, Q).$$

This proves that the function  $g_\Phi$  is Lipschitz for points  $P$  and  $Q$  satisfying  $\Delta(P, Q) \geq R$ .

Next we consider the case that  $\Delta(P, Q) < R$ . Notice the strict inequality, so in particular  $\Delta(P, Q) < 1$ . We have  $\|x\| = \|y\| = 1$  by assumption, so from Lemma 6 we can find an index  $k$  such that  $|x_k| = |y_k| = 1$ .

In order to complete the proof, we expand  $\Phi(x + h)$  as

$$\Phi(x + h) = \Phi(x) + \sum_{i=0}^N h_i B_i(x, h),$$

where each  $B_i$  is a vector of polynomials whose coefficients are linear forms (with integer coefficients) in the coefficients of  $\Phi$ . Then using

the particular index  $k$  determined above, we compute

$$\begin{aligned}
\|\Phi(x)\| &= \|y_k^d \Phi(x)\| \\
&= \|\Phi(y_k x)\| \\
&= \|\Phi(x_k y + y_k x - x_k y)\| \\
&= \left\| \Phi(x_k y) + \sum_{i=0}^N (y_k x_i - x_k y_i) B_i(x_k y, y_k x - x_k y) \right\|. \quad (11)
\end{aligned}$$

Now we observe that

$$\begin{aligned}
\left\| \sum_{i=0}^N (y_k x_i - x_k y_i) B_i(x_k y, y_k x - x_k y) \right\| &\leq \max_i |y_k x_i - x_k y_i| \\
&\leq \Delta(P, Q) < R,
\end{aligned}$$

while in the other direction we have

$$\|\Phi(x_k y)\| = |x_k|^d \|\Phi(y)\| = \|\Phi(y)\| \geq R.$$

Hence the first term in the righthand side of (11) has absolute value strictly larger than the second term, so we deduce that

$$\|\Phi(x)\| = \|\Phi(x_k y)\| = |x_k|^d \|\Phi(y)\| = \|\Phi(y)\|.$$

Hence

$$g_\Phi(P) - g_\Phi(Q) = \frac{1}{d} \log \frac{\|\Phi(x)\|}{\|\Phi(y)\|} = 0.$$

We have thus proven that if  $\Delta(P, Q) < R$ , then  $g_\Phi(P) = g_\Phi(Q)$ , which completes the proof of Theorem 13.  $\square$

### 3. ELEMENTARY PROPERTIES OF THE GREEN FUNCTION

In this section we recall from [15] the definition and basic properties of nonarchimedean Green functions. Note that what we call *nonarchimedean Green functions* are called *homogeneous local canonical height functions* in [2], and the (Arakelov) Green functions in [2] are functions on  $\mathbb{P}^1 \times \mathbb{P}^1$  with a logarithmic pole along the diagonal.

**Theorem 14.** *Let  $\varphi : \mathbb{P}_K^N \rightarrow \mathbb{P}_K^N$  be a morphism of degree  $d \geq 2$  and let  $\Phi : K^{N+1} \rightarrow K^{N+1}$  be a lift of  $\varphi$ .*

(a) *There is a unique function*

$$G_\Phi : (K^{N+1})^* \longrightarrow \mathbb{R}$$

*satisfying*

$$G_\Phi(\Phi(x)) = dG_\Phi(x) \quad \text{and} \quad G_\Phi(x) = \log \|x\| + O(1). \quad (12)$$

*The function  $G_\Phi$  is called the Green function of  $\Phi$ .*

(b) *The value of the Green function is given by the limit*

$$G_{\Phi}(x) = \lim_{n \rightarrow \infty} \frac{1}{d^n} \log \|\Phi^n(x)\|.$$

(c) *The Green function satisfies*

$$G_{\Phi}(cx) = G_{\Phi}(x) + \log |c| \quad \text{for all } c \in K^* \text{ and all } x \in (K^{N+1})^*.$$

(d) *If we use a different lift  $c\Phi$  in place of  $\Phi$ , then the Green function changes by a constant amount,*

$$G_{c\Phi}(x) = G_{\Phi}(x) + \frac{1}{d-1} \log |c|.$$

*Proof.* See [15, Theorem 7] for (a,b,c) and [15, Lemma 8] for (d).  $\square$

**Definition.** Let  $\varphi : \mathbb{P}_K^N \rightarrow \mathbb{P}_K^N$  be a morphism of degree  $d \geq 2$ , let  $\Phi$  be a lift of  $\varphi$ , and let  $G_{\Phi}$  be the associated Green function. We define the (modified) Green function of  $\varphi$  to be the function

$$\begin{aligned} \hat{g}_{\Phi} : \mathbb{P}^N(K) &\longrightarrow \mathbb{R}, \\ \hat{g}_{\Phi}(P) &= G_{\Phi}(x) - \log \|x\| \quad \text{for any } x \in \pi^{-1}(P). \end{aligned} \tag{13}$$

We end this section by proving a few elementary properties of the modified Green function.

**Proposition 15.** *Let  $\varphi : \mathbb{P}_K^N \rightarrow \mathbb{P}_K^N$  be a morphism of degree  $d \geq 2$ , let  $\Phi : K^{N+1} \rightarrow K^{N+1}$  be a lift of  $\varphi$ , and let  $\hat{g}_{\Phi}$  be the modified Green function defined by (13).*

- (a)  $\hat{g}_{\Phi}(P)$  does not depend on the choice of the lift  $x \in K^{N+1}$  of  $P$ , so  $\hat{g}_{\Phi}$  is a well-defined function on  $\mathbb{P}^N(K)$ .
- (b) Let  $g_{\Phi}(P) = d^{-1} \log \|\Phi(x)\| - \log \|x\|$  be the function defined by (7) in the statement of Theorem 13. Then

$$\hat{g}_{\Phi}(\varphi(P)) = d\hat{g}_{\Phi}(P) - dg_{\Phi}(P).$$

(c) *The Green function  $\hat{g}_{\Phi}$  is given by the series*

$$\hat{g}_{\Phi}(P) = \sum_{n=0}^{\infty} \frac{1}{d^n} g_{\Phi}(\varphi^n(P)).$$

- (d) *Assume that  $\Phi$  is a minimal lift of  $\varphi$ . Then the Green function  $\hat{g}_{\Phi}$  is nonpositive. Further,  $\hat{g}_{\Phi}(P) = 0$  if and only if  $g_{\Phi}(\varphi^n(P)) = 0$  for all  $n \geq 0$ . (See Theorem 32 for a characterization of the set where  $\hat{g}_{\Phi}(P) = 0$ .)*

*Proof.* (a) The homogeneity of the Green function (Theorem 14(c)) implies that

$$G_\Phi(cx) - \log \|cx\| = G_\Phi(x) - \log(x) \quad \text{for all } c \in K^*.$$

(b) The transformation property for  $G_\Phi$  (Theorem 14(a)) gives

$$\begin{aligned} \hat{g}_\Phi(\varphi(P)) &= G_\Phi(\Phi(x)) - \log \|\Phi(x)\| \\ &= dG_\Phi(x) - \log \|\Phi(x)\| \\ &= d\hat{g}_\Phi(P) - (\log \|\Phi(x)\| - d \log \|x\|) \\ &= d\hat{g}_\Phi(P) - dg_\Phi(P). \end{aligned}$$

(c) This follows from the usual telescoping sum argument. Thus

$$\begin{aligned} \sum_{n=0}^k \frac{1}{d^n} g_\Phi(\varphi^n(P)) &= \sum_{n=0}^k \frac{1}{d^n} \left( \frac{1}{d} \log \|\Phi^{n+1}(x)\| - \log \|\Phi^n(x)\| \right) \\ &= \frac{1}{d^k} \log \|\Phi^k(x)\| - \log \|x\|. \end{aligned}$$

Letting  $k \rightarrow \infty$ , the righthand side goes to  $\hat{g}_\Phi(P)$ .

(d) The upper bound in Proposition 9 tells us that the function  $g_\Phi$  satisfies

$$g_\Phi(P) = \frac{1}{d} \log \|\Phi(x)\| - \log \|x\| \leq 0.$$

Hence the sum in (c) consists entirely of nonpositive terms. It follows that  $\hat{g}_\Phi(P) \leq 0$ , and further  $\hat{g}_\Phi(P) = 0$  if and only if every term in the sum vanishes.  $\square$

*Remark 16.* Chambert-Loir tells us that the modified Green function  $\hat{g}_\Phi$  is related to the canonical [admissible] metric on the line bundle  $\mathcal{O}_{\mathbb{P}^N}(1)$  introduced by Zhang [24].

Precisely, we write  $\|\cdot\|_{\text{sup}}$  for the metric on  $\mathcal{O}_{\mathbb{P}^N}(1)$  defined by

$$\|s\|_{\text{sup}}(P) = \frac{|s(x)|}{\|x\|} \quad \text{for } s \in \Gamma(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1)) \text{ and any } x \in \pi^{-1}(P),$$

and  $\|\cdot\|_\Phi$  for the canonical metric on  $\mathcal{O}_{\mathbb{P}^N}(1)$  associated to  $\varphi : \mathbb{P}^N \rightarrow \mathbb{P}^N$  and a lift  $\Phi$  of  $\varphi$  (see [24, Theorem (2.2)]). Then we obtain

$$\hat{g}_\Phi = \log \frac{\|\cdot\|_{\text{sup}}}{\|\cdot\|_\Phi}.$$

Hence properties of  $\hat{g}_\Phi$  give the corresponding properties of the canonical metric  $\|\cdot\|_\Phi$ .

## 4. HÖLDER CONTINUITY OF THE GREEN FUNCTION

Our goal in this section is to prove that  $\hat{g}_\Phi$  is Hölder continuous on  $\mathbb{P}^N$ . We follow the argument of Dinh–Sibony [10, Proposition 2.4] (See also Favre–Riviera–Letelier [11, Proposition 6.5]). Over a nonarchimedean valuation field, we easily obtain explicit constants for Hölder continuity. We begin with an elementary lemma.

**Lemma 17.** *Let  $a, b, D$  be constants satisfying  $a > 1$ ,  $b > 1$  and  $0 < D \leq 1$ . Then*

$$\min\{Da^k + b^{-k} : k \in \mathbb{Z}, k > 0\} \leq 2aD^{\frac{\log b}{\log ab}}.$$

*Proof.* Let  $t \in \mathbb{R}$  be the number

$$t = \frac{\log(D^{-1})}{\log ab}.$$

Then the assumptions on  $a, b, D$  imply that  $t \geq 0$ , and by definition of  $t$  we have  $Da^t = b^{-t}$ . Hence

$$Da^t + b^{-t} = 2Da^{\frac{\log(D^{-1})}{\log ab}} = 2D \cdot D^{-\frac{\log a}{\log ab}} = 2D^{\frac{\log b}{\log ab}}.$$

We put  $k = \lfloor t \rfloor + 1$ . Then  $k$  is a positive integer and we have

$$\begin{aligned} Da^k + b^{-k} &= a^{k-t}Da^t + b^{-(k-t)}b^{-t} \leq aDa^t + b^{-t} \\ &\leq a(Da^t + b^{-t}) = 2aD^{\frac{\log b}{\log ab}}. \end{aligned}$$

This completes the proof of the lemma.  $\square$

We now prove that the nonarchimedean Green function is Hölder continuous and give explicit Hölder constants.

**Theorem 18.** *The modified Green function  $\hat{g}_\Phi : \mathbb{P}^N(K) \rightarrow \mathbb{R}$  defined by (13) is Hölder continuous. More precisely, let*

$$u = u(\varphi) = \max\{2d, |\text{Res}(\varphi)|^{-2}\}.$$

*Then*

$$|\hat{g}_\Phi(P) - \hat{g}_\Phi(Q)| \leq \frac{2u \log u}{d} \Delta(P, Q)^{\frac{\log d}{\log u}} \quad \text{for all } P, Q \in \mathbb{P}^N(K). \quad (14)$$

*Proof.* In general, the Green function  $G_\Phi$  and the modified Green function  $\hat{g}_\Phi$  depend on the chosen lift  $\Phi$  of  $\varphi$ . However, Theorem 14(d) tells us that  $G_{c\Phi} - G_\Phi$  is constant, so the difference  $\hat{g}_\Phi(P) - \hat{g}_\Phi(Q)$  is independent of the chosen lift  $\Phi$  of  $\varphi$ . Hence without loss of generality we may assume that  $\Phi$  is a minimal lift of  $\varphi$ .

To ease notation, we let  $R = |\text{Res}(\varphi)|$  as usual. We also recall the function

$$g_\Phi(P) = \frac{1}{d} \log \|\Phi(x)\| - \log \|x\|$$

used in Theorem 13. Note that Proposition 9 tells us that  $g_\Phi$  is a bounded function,

$$\frac{\log(R)}{d} \leq g(P) \leq 0 \quad \text{for all } P \in \mathbb{P}^N(K). \quad (15)$$

Further, Proposition 15(c) says that we can write  $\hat{g}_\Phi$  as a telescoping sum,

$$\hat{g}_\Phi(P) = \sum_{n=0}^{\infty} \frac{1}{d^n} g_\Phi(\varphi^n(P)).$$

Let  $k$  be an auxiliary integer to be chosen later. We compute

$$\begin{aligned} & |\hat{g}_\Phi(P) - \hat{g}_\Phi(Q)| \\ &= \left| \sum_{n=0}^{\infty} \frac{1}{d^n} (g_\Phi(\varphi^n(P)) - g_\Phi(\varphi^n(Q))) \right| \\ &\leq \sum_{n=0}^{k-1} \frac{1}{d^n} |g_\Phi(\varphi^n(P)) - g_\Phi(\varphi^n(Q))| + 2 \left( \sum_{n=k}^{\infty} \frac{1}{d^n} \right) \sup_{T \in \mathbb{P}^N(K)} |g_\Phi(T)| \\ &\leq \sum_{n=0}^{k-1} \frac{1}{d^n} \cdot \frac{\log(R^{-1})}{dR} \Delta(\varphi^n(P), \varphi^n(Q)) + \frac{2}{d^k} \cdot \frac{1}{1-d^{-1}} \cdot \frac{\log(R^{-1})}{d} \\ &\quad \text{from Theorem 13 and (15),} \\ &\leq \sum_{n=0}^{k-1} \frac{1}{d^n} \cdot \frac{\log(R^{-1})}{dR} \cdot R^{-2n} \Delta(P, Q) + \frac{2 \log(R^{-1})}{d-1} \cdot \frac{1}{d^k} \\ &\quad \text{from Theorem 10,} \\ &\leq 2 \log(R^{-1}) \cdot \left( \frac{\Delta(P, Q)}{2} \cdot \sum_{n=1}^k \frac{1}{(dR^2)^n} + \frac{1}{d^k} \right). \end{aligned} \quad (16)$$

The most interesting case is when  $dR^2$  is small, say  $dR^2 \leq \frac{1}{2}$ , so we consider that case first. Then the bound (16) yields

$$|\hat{g}_\Phi(P) - \hat{g}_\Phi(Q)| \leq 2 \log(R^{-1}) \cdot \left( \Delta(P, Q) \left( \frac{1}{dR^2} \right)^k + d^{-k} \right).$$

We now choose  $k$  as described in Lemma 17. This gives the desired upper bound

$$|\hat{g}_\Phi(P) - \hat{g}_\Phi(Q)| \leq \frac{4 \log(R^{-1})}{dR^2} \cdot \Delta(P, Q)^{\frac{\log d}{\log(R^{-2})}} \quad (17)$$

Next we suppose that  $dR^2 \geq \frac{1}{2}$ . Then  $\sum_{n=1}^k (dR^2)^{-n} < 2^{k+1}$ , so (16) gives

$$|\hat{g}_\Phi(P) - \hat{g}_\Phi(Q)| \leq \log 2d \cdot (\Delta(P, Q)2^k + d^{-k}).$$

Now another application of Lemma 17 yields the upper bound

$$|\hat{g}_\Phi(P) - \hat{g}_\Phi(Q)| \leq 4 \log 2d \cdot \Delta(P, Q)^{\frac{\log d}{\log 2d}}. \quad (18)$$

Combining (17) and (18) completes the proof that  $\hat{g}_\Phi$  is Hölder continuous with the explicit constants listed in (14).  $\square$

## 5. DISTANCE FUNCTIONS AND THE OPEN MAPPING PROPERTY

In this section we recall a distribution relation for distance functions proven in [22], where it was used to prove a quantitative nonarchimedean inverse function theorem. We apply the distribution relation to give a short proof that finite morphisms  $\varphi : \mathbb{P}^N \rightarrow \mathbb{P}^N$  over nonarchimedean fields are open maps, i.e., they map open sets to open sets. More generally, the same is true for any finite morphism of projective varieties.

**Proposition 19** (Distribution Relation). *Let  $\varphi : \mathbb{P}^N \rightarrow \mathbb{P}^N$  be a morphism of degree  $d \geq 1$  defined over  $K$  and let  $P, T \in \mathbb{P}^N(K)$ . Then*

$$-\log \Delta(\varphi(P), T) = \sum_{Q \in \varphi^{-1}(T)} -e_\varphi(Q) \log \Delta(P, Q) + O_\varphi(1),$$

where  $e_\varphi(Q)$  is the ramification index of  $\varphi$  at  $Q$  and the big- $O$  constant depends on  $\varphi$ , but is independent of  $P$  and  $Q$ .

In particular, there is a constant  $c = c(\varphi) \geq 1$  such that for all  $P, T \in \mathbb{P}^N(K)$  we have

$$\min_{Q \in \varphi^{-1}(T)} \Delta(P, Q) \leq c \Delta(\varphi(P), T)^{1/d}. \quad (19)$$

*Proof.* The first statement is a special case of [22, Proposition 6.2(b)]. Note that since  $\mathbb{P}^N$  is projective and  $\varphi$  is defined on all of  $\mathbb{P}^N$ , we do not need the  $\lambda_{\partial W \times V}$  term that appears in [22]. The second statement is immediate from exponentiating the first statement and using the fact that  $\sum_{Q \in \varphi^{-1}(T)} e_\varphi(Q) = d$ .  $\square$

*Remark 20.* For refined calculations, there is a version of (19) without the  $1/d$  exponent provided that  $P$  is not in the ramification locus of  $\varphi$ . More precisely, [22, Theorem 6.1] implies that if  $\varphi$  is unramified at  $P \in \mathbb{P}^N(K)$ , then there is a disk  $D_r(P)$  around  $P$  such that the map

$$\varphi : D_r(P) \longrightarrow \varphi(D_r(P))$$



is bijective and biLipschitz, i.e., both  $\varphi$  and  $\varphi^{-1}$  are Lipschitz. Of course, we have already seen that  $\varphi$  is Lipschitz (Theorem 10), the new information is that  $\varphi^{-1}$  is also Lipschitz. Notice that even if  $\varphi$  is ramified at  $P$ , Proposition 19 more-or-less says that  $\varphi^{-1}$  (which doesn't quite exist) satisfies  $\Delta(\varphi^{-1}(P), \varphi^{-1}(Q)) \ll \Delta(P, Q)^{1/d}$ , so  $\varphi^{-1}$  is locally Hölder continuous.

**Corollary 21.** *Let  $\varphi : \mathbb{P}^N \rightarrow \mathbb{P}^N$  be a morphism of degree  $d \geq 1$  defined over  $K$ . Then  $\varphi$  is an open mapping, i.e.,  $\varphi$  maps open sets to open sets.*

*Proof.* Let  $U \subset \mathbb{P}^N(K)$  be an open set and let  $\varphi(P) \in \varphi(U)$  be a point in the image of  $\varphi$ . We need to find a disk around  $\varphi(P)$  that is contained in  $\varphi(U)$ . Since  $U$  is open, we can find an  $\epsilon > 0$  so that  $D_\epsilon(P) \subset U$ . Let  $\delta = (\epsilon/c)^d$ , where  $c$  is the constant appearing in (19) in Proposition 19. We claim that  $D_\delta(\varphi(P)) \subset \varphi(U)$ , which will complete the proof.

So let  $T \in D_\delta(\varphi(P))$ . We apply the second statement in Proposition 19 to find a point  $Q \in \varphi^{-1}(T)$  satisfying

$$\Delta(P, Q) \leq c\Delta(\varphi(P), T)^{1/d} < c\delta^{1/d} = \epsilon.$$

Hence  $Q \in D_\epsilon(P) \subset U$ , so  $T = \varphi(Q) \in \varphi(U)$ .  $\square$

We note that the Hölder-type inequality (19) that follows from the distribution relation (Proposition 19) can be used to prove directly from the definition that the Fatou and Julia sets of  $\varphi$  are completely invariant. However, since we have not yet defined the Fatou and Julia sets, we defer the proof until Section 8, where we instead give a short proof based on our characterization of the Fatou set as the set on which the Green function is locally constant.

## 6. NONARCHIMEDEAN ANALYSIS

Let  $K$  be an algebraically closed field that is complete with respect to a nonarchimedean absolute value as usual. In this section, we recall some basic facts from nonarchimedean analysis. For details we refer the reader to [8].

Let  $a = (a_1, \dots, a_N) \in K^N$  and let  $r \in |K^*|$  be a real number in the value group of  $K$ . A formal power series

$$\Psi(x) = \sum_{i_1, \dots, i_N \geq 0} c_{i_1 \dots i_N} (x_1 - a_1)^{i_1} \cdots (x_N - a_N)^{i_N}$$

is said to be *analytic on  $\bar{B}(a, r)$*  if the coefficients  $c_{i_1 \dots i_N} \in K$  satisfy

$$\lim_{i_1 + \dots + i_N \rightarrow \infty} |c_{i_1 \dots i_N}| r^{i_1 + \dots + i_N} = 0.$$

Then  $\Psi(x)$  defines a function  $\Psi : \bar{B}(a, r) \rightarrow K$ . The *Gauss norm* of  $\Psi$  on  $\bar{B}(a, r)$  is the quantity

$$\|\Psi\|_{\bar{B}(a, r)} = \sup_{i_1 \dots i_N} \{ |c_{i_1 \dots i_N}| r^{i_1 + \dots + i_N} \}.$$

If  $\Psi$  is analytic on  $\bar{B}(a, r)$ , then  $\|\Psi\|_{\bar{B}(a, r)}$  is finite, and the strong triangle inequality gives

$$|\Psi(x)| \leq \|\Psi\|_{\bar{B}(a, r)} \quad \text{for all } x \in \bar{B}(a, r).$$

**Lemma 22.** *Let  $\Psi$  be an analytic function on  $\bar{B}(a, r)$ .*

(a) [Maximum Principle] *There is an  $x' \in \bar{B}(a, r)$  such that*

$$|\Psi(x')| = \|\Psi\|_{\bar{B}(a, r)}.$$

(b) *For all  $x, y \in \bar{B}(a, r)$ , we have*

$$|\Psi(x) - \Psi(y)| \leq \frac{\|\Psi\|_{\bar{B}(a, r)}}{r} \|x - y\|.$$

*Proof.* We fix a  $b \in K^*$  with  $|b| = r$ .

(a) For a proof when  $\bar{B}(a, r)$  is the unit polydisk, i.e.,  $a = 0$  and  $r = 1$ , see [8, § 5.1.4, Propositions 3 and 4]. As in [13, Proposition 1.1], the general case follows using the isomorphism

$$\bar{B}(a, r) \longrightarrow \bar{B}(0, 1), \quad x \longmapsto \frac{x - a}{b}.$$

(b) To ease notation, we let  $I = (i_1, \dots, i_N)$  and write  $(x - a)^I$  for the product  $\prod_{j=1}^N (x_j - a_j)^{i_j}$ . Similarly for  $(y - a)^I$  and  $r^I = r^{i_1 + \dots + i_N}$ . Then

$$\begin{aligned} |\Psi(x) - \Psi(y)| &= \left| \sum_I c_I ((x - a)^I - (y - a)^I) \right| \\ &\leq \sup_I |c_I| \cdot |(x - a)^I - (y - a)^I| \\ &\leq \left( \sup_I |c_I| r^I \right) \cdot \sup_I \left| \left( \frac{x - a}{b} \right)^I - \left( \frac{y - a}{b} \right)^I \right| \\ &= \|\Psi\|_{\bar{B}(a, r)} \cdot \sup_I \left| \left( \frac{x - a}{b} \right)^I - \left( \frac{y - a}{b} \right)^I \right|. \end{aligned}$$

We now use the fact that for all  $I$  and  $j$  there exist polynomials  $F_{I,j}(X, Y) \in \mathbb{Z}[X, Y]$  such that

$$X^I - Y^I := \left( \prod_{j=1}^N X_j^{i_j} \right) - \left( \prod_{j=1}^N Y_j^{i_j} \right) = \sum_{j=1}^N (X_j - Y_j) F_{I,j}(X, Y).$$

Putting  $X = (x - a)/b$  and  $Y = (y - a)/b$  and using the triangle inequality yields

$$\left| \left( \frac{x - a}{b} \right)^I - \left( \frac{y - a}{b} \right)^I \right| \leq \max_{1 \leq j \leq N} \left| \frac{x_j - y_j}{b} \right| \cdot \left| F_{I,j} \left( \frac{x - a}{b}, \frac{y - a}{b} \right) \right|.$$

We know that  $|b| = r$  and  $x, y \in \bar{B}(a, r)$ , and also  $F_{I,j}$  has integer coefficients, so  $|F_{I,j}((x - a)/b, (y - a)/b)| \leq 1$ . Hence

$$|\Psi(x) - \Psi(y)| \leq \|\Psi\|_{\bar{B}(a, r)} \cdot \max_{1 \leq j \leq N} \left| \frac{x_j - y_j}{b} \right| = \|\Psi\|_{\bar{B}(a, r)} \cdot \frac{\|x - y\|}{r}.$$

□

**Lemma 23.** *Let  $\mathcal{A}$  be a family of analytic functions on  $\bar{B}(a, r)$ . Assume that there is a constant  $C > 0$  such that*

$$|\Psi(x)| \leq C \quad \text{for all } x \in \bar{B}(a, r) \text{ and all } \Psi \in \mathcal{A}.$$

*Then for all  $x, y \in \bar{B}(a, r)$  and all  $\Psi, \Lambda \in \mathcal{A}$  we have*

$$|\Psi(x)\Lambda(y) - \Psi(y)\Lambda(x)| \leq \frac{C^2}{r} \|x - y\|.$$

*Proof.* By Lemma 22(a), we have  $\|\Psi\|_{\bar{B}(a, r)} \leq C$  for all  $\Psi \in \mathcal{A}$ . Then for any  $x, y \in \bar{B}(a, r)$  we have

$$\begin{aligned} & |\Psi(x)\Lambda(y) - \Psi(y)\Lambda(x)| \\ &= |\Lambda(y)(\Psi(x) - \Psi(y)) - \Psi(y)(\Lambda(x) - \Lambda(y))| \\ &\leq \max\{|\Lambda(y)| \cdot |\Psi(x) - \Psi(y)|, |\Psi(y)| \cdot |\Lambda(x) - \Lambda(y)|\} \\ &\leq \max\{\|\Lambda\|_{\bar{B}(a, r)} |\Psi(x) - \Psi(y)|, \|\Psi\|_{\bar{B}(a, r)} |\Lambda(x) - \Lambda(y)|\} \\ &\leq \frac{\|\Psi\|_{\bar{B}(a, r)} \cdot \|\Lambda\|_{\bar{B}(a, r)}}{r} \|x - y\| \quad \text{from Lemma 22(b)} \\ &\leq \frac{C^2}{r} \|x - y\|. \end{aligned}$$

□

## 7. THE FATOU AND JULIA SETS

In this section we recall the definition of the Fatou and Julia sets for a family of maps on a metric space. In our case, the metric space is  $\mathbb{P}^N(K)$  with the metric induced by the chordal distance function  $\Delta$ .

**Definition.** Let  $U$  be an open subset of  $\mathbb{P}^N(K)$ . A family of maps  $\mathcal{A}$  from  $U$  to  $\mathbb{P}^N(K)$  is *equicontinuous* at a point  $P \in U$  if for every  $\epsilon > 0$  there is a  $\delta > 0$  such that

$$\psi(D_\delta(P)) \subset D_\epsilon(\psi(P)) \quad \text{for all } \psi \in \mathcal{A}.$$

We note that the open disks  $D_\delta(P)$  and  $D_\epsilon(\psi(P))$  may be replaced by closed disks  $\bar{D}_\delta(P)$  and  $\bar{D}_\epsilon(\psi(P))$  without affecting the definition. The family  $\mathcal{A}$  is *equicontinuous on  $U$*  if it is equicontinuous at every  $P \in U$ .

We note that in general, equicontinuity at a point  $P$  is not an open condition, since  $\delta$  may depend on both  $\epsilon$  and  $P$ . In particular, it is weaker than the related property of  $\mathcal{A}$  being *uniformly continuous on  $U$* , in which a single  $\delta$  is required to work for every  $P \in U$ .

The family  $\mathcal{A}$  is called *(locally) uniformly Lipschitz at  $P \in U$*  if there exists a constant  $C = C(\mathcal{A}, P)$  and a radius  $r = r(\mathcal{A}, P)$  such that

$$\Delta(\psi(Q), \psi(R)) \leq C\Delta(Q, R) \quad \text{for all } Q, R \in \bar{D}_r(P) \text{ and all } \psi \in \mathcal{A}.$$

In other words,  $\mathcal{A}$  is locally uniformly Lipschitz at  $P$  if each map in  $\mathcal{A}$  is Lipschitz in some neighborhood of  $P$  and further there is a single Lipschitz constant that works for every  $\psi \in \mathcal{A}$ .

If the family  $\mathcal{A}$  is equicontinuous on each open subsets  $U_\alpha$  of  $\mathbb{P}(K)$ , then it is equicontinuous on the union  $\bigcup_\alpha U_\alpha$ . Taking collections  $\{U_\alpha\}$  to be all open subsets of  $\mathbb{P}^N(K)$  on which  $\mathcal{A}$  is equicontinuous, we are led to the following definition.

For convenience, we say that a map  $\varphi : \mathbb{P}^N \rightarrow \mathbb{P}^N$  is equicontinuous if the family of iterates  $\{\varphi^n\}_{n \geq 1}$  is equicontinuous, and similarly  $\varphi$  is locally uniformly Lipschitz if its iterates are.

**Definition.** Let  $\varphi : \mathbb{P}_K^N \rightarrow \mathbb{P}_K^N$  be a morphism. The *Fatou set* of  $\varphi$ , denoted  $\mathcal{F}(\varphi)$ , is the union of all open subsets of  $\mathbb{P}^N(K)$  on which  $\varphi$  is equicontinuous. Equivalently, the Fatou set  $\mathcal{F}(\varphi)$  is the largest open set such that the family  $\{\varphi^n\}_{n=1}^\infty$  is equicontinuous at every point of  $\mathcal{F}(\varphi)$ . The *Julia set* of  $\varphi$ , denoted  $\mathcal{J}(\varphi)$ , is the complement of  $\mathcal{F}(\varphi)$ . Thus by definition the Fatou set is open and the Julia set is closed.

## 8. THE GREEN FUNCTION ON THE FATOU AND JULIA SETS

In this section we characterize the Fatou set of  $\varphi$  as the set on which the (modified) Green function  $\hat{g}_\Phi$  is locally constant. Along the way, we prove that  $\varphi$  is locally uniformly Lipschitz on the Fatou set.

**Theorem 24.** *Let  $\varphi : \mathbb{P}_K^N \rightarrow \mathbb{P}_K^N$  be a morphism of degree  $d \geq 2$  as usual, let  $\Phi$  be a lift of  $\varphi$ , let  $\hat{g}_\Phi$  be the (modified) Green function defined by (1) and (2), and let  $P \in \mathbb{P}^N(K)$ . Then the following are equivalent:*

- (a) *The iterates of  $\varphi$  are equicontinuous at every point in some neighborhood of  $P$ , i.e.,  $P \in \mathcal{F}(\varphi)$ .*
- (b) *The iterates of  $\varphi$  are locally uniformly Lipschitz at  $P$ .*
- (c) *The function  $\hat{g}_\Phi$  is constant on a neighborhood of  $P$ .*

*Proof.* It is clear that being locally uniformly Lipschitz at  $P$  is stronger than being equicontinuous in a neighborhood of  $P$ , so (b) implies (a).

Next we show that (a) implies (c), so we let  $P \in \mathcal{F}(\varphi)$  and let  $g_\Phi$  be the usual function

$$g_\Phi(Q) = \frac{1}{d} \log \|\Phi(y)\| - \log \|y\| \quad \text{for } y \in \pi^{-1}(Q)$$

as in Theorem 13. We take  $\epsilon = \frac{1}{2}|\text{Res}(\varphi)|$  in the definition of equicontinuity and find a  $\delta = \delta(\epsilon, P) > 0$  so that

$$\Delta(P, Q) \leq \delta \implies \Delta(\varphi^n(P), \varphi^n(Q)) \leq \epsilon < |\text{Res}(\varphi)|$$

for all  $Q$  and all  $n \geq 0$ .

It follows from Theorem 13 that

$$\Delta(P, Q) \leq \delta \implies g_\Phi(\varphi^n(P)) = g_\Phi(\varphi^n(Q)) \quad \text{for all } Q \text{ and all } n \geq 0.$$

Then the series representation of  $\hat{g}_\Phi$  given in Proposition 15(c) implies that

$$\hat{g}_\Phi(P) = \hat{g}_\Phi(Q) \quad \text{for all } Q \in \bar{D}_\delta(P).$$

Hence  $\hat{g}_\Phi$  is constant on  $\bar{D}_\delta(P)$ , which completes the proof that (a) implies (c).

It remains to show that (c) implies (b). So we assume that  $\hat{g}_\Phi$  is constant on  $\bar{D}_\delta(P)$  and need to prove that the iterates of  $\varphi$  are Lipschitz on  $\bar{D}_\delta(P)$  with a uniform Lipschitz constant. We choose a minimal lift  $\Phi : (K^{N+1})^* \rightarrow (K^{N+1})^*$  of  $\varphi$  and define functions  $g_{\Phi,n}$  by

$$g_{\Phi,n}(Q) = \frac{1}{d^n} \log \|\Phi^n(y)\| - \log \|y\| \quad \text{for } Q \in \mathbb{P}^N(K) \text{ and } y \in \pi^{-1}(Q).$$

Then as in the proof of Proposition 15, we can use a telescoping sum to write

$$\begin{aligned} \hat{g}_\Phi(Q) - g_{\Phi,n}(Q) &= \lim_{k \rightarrow \infty} \frac{1}{d^k} \log \|\Phi^k(y)\| - \frac{1}{d^n} \log \|\Phi^n(y)\| \\ &= \sum_{k=n}^{\infty} \left( \frac{1}{d^{k+1}} \log \|\Phi^{k+1}(y)\| - \frac{1}{d^k} \log \|\Phi^k(y)\| \right). \end{aligned}$$

Then Proposition 9 gives the estimate

$$\begin{aligned} |\hat{g}_\Phi(Q) - g_{\Phi,n}(Q)| &\leq \sum_{k \geq n} \frac{1}{d^k} \left| \frac{1}{d} \log \|\Phi^{k+1}(y)\| - \log \|\Phi^k(y)\| \right| \\ &\leq \sum_{k \geq n} \frac{1}{d^{k+1}} \log |\text{Res}(\varphi)|^{-1} = \frac{C_1}{d^n}, \end{aligned} \tag{20}$$

where for convenience we let  $C_1 = \frac{1}{d-1} \log |\text{Res}(\varphi)|^{-1}$ . (In particular, the constant  $C_1$  only depends on  $\varphi$ .)

Recall that we have fixed a point  $P \in \mathbb{P}^N(K)$ . It would be convenient if we would find an element  $h \in K^*$  satisfying  $\log |h| = \hat{g}_\Phi(P)$ , but even if  $K = \mathbb{C}_p$ , we only have  $|\mathbb{C}_p^*| = p^\mathbb{Q}$ . However,  $\log(p^\mathbb{Q})$  is dense in  $\mathbb{R}$ , so we can find a sequence of elements  $h_n \in K^*$  satisfying

$$|\hat{g}_\Phi(P) - \log |h_n|| \leq \frac{1}{d^n} \quad \text{for all } n \geq 0. \quad (21)$$

Now let  $Q \in \bar{D}_\delta(P)$  and choose lifts  $x \in \pi^{-1}(P)$  and  $y \in \pi^{-1}(Q)$ . Note that  $\hat{g}_\Phi(Q) = \hat{g}_\Phi(P)$ , since by assumption  $\hat{g}_\Phi$  is constant on  $\bar{D}_\delta(P)$ . This allows us to estimate

$$\begin{aligned} \left| \frac{1}{d^n} \log \|h_n^{-d^n} \Phi^n(y)\| - \log \|y\| \right| &= |g_{\Phi,n}(Q) - \log |h_n|| \\ &= |g_{\Phi,n}(Q) - \hat{g}_\Phi(Q) + \hat{g}_\Phi(P) - \log |h_n|| \\ &\leq |g_{\Phi,n}(Q) - \hat{g}_\Phi(Q)| + |\hat{g}_\Phi(P) - \log |h_n|| \\ &\leq \frac{C_1 + 1}{d^n} \quad \text{from (20) and (21).} \end{aligned}$$

Hence if we define a new sequence of functions  $(\Lambda_{\Phi,n})_{n \geq 0}$  by the formula

$$\Lambda_{\Phi,n}(y) = h_n^{-d^n} \Phi^n(y)$$

and a new constant  $C_2 = e^{C_1+1}$ , then these new functions satisfy

$$C_2^{-1} \leq \frac{\|\Lambda_{\Phi,n}(y)\|}{\|y\|^{d^n}} \leq C_2 \quad \text{for all } \pi(y) \in \bar{D}_\delta(P) \text{ and } n \geq 0. \quad (22)$$

Notice that  $\Lambda_{\Phi,n}$  is also a lift of  $\varphi^n$ , since we have merely multiplied  $\Phi^n$  by a constant.

Reordering the coordinates if necessary and dividing by the largest one, we may assume without loss of generality that  $x \in \pi^{-1}(P)$  satisfies  $x_0 = 1 = \|x\|$ . Thus if we let  $a = (x_1, \dots, x_N)$ , then Lemma 7(b) says that there is an isometric isomorphism

$$\sigma : \bar{B}(a, \delta) \longrightarrow \bar{D}_\delta(P), \quad \sigma(b_1, \dots, b_N) = (1 : b_1 : \dots : b_N). \quad (23)$$

Let

$$\Psi_{\Phi,n}(b_1, \dots, b_N) = \Lambda_{\Phi,n}(1, b_1, \dots, b_N)$$

be the dehomogenization of  $\Lambda_{\Phi,n}$ . Then (22) gives

$$C_2^{-1} \leq \|\Psi_{\Phi,n}(b)\| \leq C_2 \quad \text{for all } b \in \bar{B}(a, \delta) \text{ and } n \geq 0.$$

Write the coordinate functions of  $\Psi_{\Phi,n}$  as  $\Psi_{\Phi,n} = (\Psi_{n0}, \dots, \Psi_{nN})$  and consider the family of functions

$$\{\Psi_{ni} : 0 \leq i \leq N \text{ and } n \geq 0\}.$$

Every function in this family satisfies  $|\Psi_{ni}(b)| \leq C_2$  for all  $b \in \bar{B}(a, \delta)$ , so Lemma 23 tells us that

$$|\Psi_{ni}(b)\Psi_{nj}(b') - \Psi_{ni}(b')\Psi_{nj}(b)| \leq \frac{C_2^2}{\delta} \|b - b'\|$$

for all  $b, b' \in \bar{B}(a, \delta)$ , all  $0 \leq i, j \leq N$ , and all  $n \geq 0$ .

Combining this with the lower bound  $\|\Psi_{\Phi, n}(b)\| \geq C_2^{-1}$  yields

$$\frac{|\Psi_{ni}(b)\Psi_{nj}(b') - \Psi_{ni}(b')\Psi_{nj}(b)|}{\|\Psi_{\Phi, n}(b)\| \cdot \|\Psi_{\Phi, n}(b')\|} \leq \frac{C_2^4}{\delta} \|b - b'\|$$

for all  $b, b' \in \bar{B}(a, \delta)$ , all  $0 \leq i, j \leq N$ , and all  $n \geq 0$ .

Now we take the maximum over all  $0 \leq i, j \leq N$  and use the definition of the chordal distance and the isometry (23). This gives

$$\Delta(\sigma(\Psi_{\Phi, n}(b)), \sigma(\Psi_{\Phi, n}(b'))) \leq \frac{C_2^4}{\delta} \Delta(\sigma(b), \sigma(b'))$$

for all  $b, b' \in \bar{B}(a, \delta)$  and all  $n \geq 0$ .

From the definitions we have  $\sigma(\Psi_{\Phi, n}(b)) = \varphi^n(\sigma(b))$  and similarly for  $b'$ , so letting  $\sigma(b) = Q$  and  $\sigma(b') = R$ , we have proven that

$$\Delta(\varphi^n(Q), \varphi^n(R)) \leq \frac{C_2^4}{\delta} \Delta(Q, R) \quad \text{for all } Q, R \in \bar{D}_\delta(P) \text{ and all } n \geq 0.$$

Hence the iterates of  $\varphi$  are uniformly Lipschitz on the disk  $\bar{D}_\delta(P)$ , since the Lipschitz constant  $C_2^4/\delta$  depends only on  $P$  and  $\varphi$ .  $\square$

Theorem 24 has a number of useful corollaries. We note that it is possible to prove these corollaries directly from the definition of the Fatou set, but the use of the Green function simplifies and unifies the proofs. The first is actually a restatement of part of Theorem 24, but we feel that it is sufficiently important to merit the extra attention. This is particularly true because some authors define the nonarchimedean Fatou set in terms of equicontinuity and others define it in terms of uniform continuity. The following corollary shows that the two definitions are equivalent, and indeed they are also equivalent to the stronger locally uniformly Lipschitz property.

**Corollary 25.** *Let  $\varphi : \mathbb{P}_K^N \rightarrow \mathbb{P}_K^N$  be a morphism of degree  $d \geq 2$ . Then  $\{\varphi^n\}_{n \geq 0}$  is locally uniformly Lipschitz on its Fatou set  $\mathcal{F}(\varphi)$ . In other words, for every  $P \in \mathcal{F}(\varphi)$  there exists a  $\delta = \delta(\varphi, P) > 0$  and a constant  $C = C(\varphi, P)$  so that*

$$\Delta(\varphi^n(Q), \varphi^n(R)) \leq C \Delta(Q, R) \quad \text{for all } Q, R \in \bar{D}_\delta(P) \text{ and all } n \geq 0.$$

*Proof.* This is the implication (a)  $\implies$  (b) in Theorem 24.  $\square$

The complete invariance of the Fatou and Julia sets is also an easy corollary of Theorem 24 and the fact that  $\varphi$  is an open mapping.

**Corollary 26.** *The Fatou set  $\mathcal{F}(\varphi)$  and the Julia set  $\mathcal{J}(\varphi)$  are completely invariant under  $\varphi$ .*

*Proof.* Since the Julia set is the complement of the Fatou set, it suffices to prove the invariance of  $\mathcal{F}(\varphi)$  under  $\varphi$  and  $\varphi^{-1}$ .

Let  $P \in \varphi^{-1}(\mathcal{F}(\varphi))$ . Theorem 24 says that the Green function  $\hat{g}_\Phi$  is constant on some disk  $\bar{D}_\epsilon(\varphi(P))$ . Since  $\varphi$  is continuous, we can find a  $\delta$  satisfying

$$0 < \delta < |\text{Res}(\varphi)| \quad \text{and} \quad \varphi(\bar{D}_\delta(P)) \subset \bar{D}_\epsilon(\varphi(P)).$$

Then by assumption,  $\hat{g}_\Phi$  is constant on the set  $\varphi(\bar{D}_\delta(P))$ . We claim that  $\hat{g}_\Phi$  is constant on  $\bar{D}_\delta(P)$ .

Proposition 15 tells us that the Green function  $\hat{g}_\Phi$  satisfies the transformation property

$$\hat{g}_\Phi(Q) = \frac{1}{d} \hat{g}_\Phi(\varphi(Q)) + g_\Phi(Q), \quad (24)$$

where  $g_\Phi$  is the function defined in Theorem 13. And we know that the function  $\hat{g}_\Phi \circ \varphi$  is constant on  $\bar{D}_\delta(P)$ . But Theorem 13 says that  $g_\Phi$  is also constant on that disk since we have chosen  $\delta < |\text{Res}(\varphi)|$ . This proves that  $\hat{g}_\Phi$  is constant in a neighborhood of  $P$ , so Theorem 24 tells us that  $P \in \mathcal{F}(\varphi)$ . Hence  $\varphi^{-1}(\mathcal{F}(\varphi)) \subseteq \mathcal{F}(\varphi)$ .

For the other direction, let  $P \in \mathcal{F}(\varphi)$ . Theorem 24 says that we can find a  $0 < \delta < |\text{Res}(\varphi)|$  such that  $\hat{g}_\Phi$  is constant on  $\bar{D}_\delta(P)$ . Since  $\varphi$  is an open mapping (Corollary 21), there is an  $\epsilon > 0$  satisfying

$$\bar{D}_\epsilon(\varphi(P)) \subset \varphi(\bar{D}_\delta(P)).$$

We claim that  $\hat{g}_\Phi$  is constant on  $\bar{D}_\epsilon(\varphi(P))$ .

For any  $Q \in \bar{D}_\epsilon(\varphi(P))$ , we write  $Q = \varphi(R)$  with  $R \in \bar{D}_\delta(P)$  and use the transformation formula (24) to compute

$$\hat{g}_\Phi(Q) = \hat{g}_\Phi(\varphi(R)) = d\hat{g}_\Phi(R) - dg_\Phi(R).$$

The function  $\hat{g}_\Phi$  is constant on  $\bar{D}_\delta(P)$ , and since  $\delta < |\text{Res}(\varphi)|$ , Theorem 13 tells us that  $g_\Phi$  is also constant on  $\bar{D}_\delta(P)$ . Hence  $\hat{g}_\Phi$  is constant on  $\bar{D}_\epsilon(\varphi(P))$ , so Theorem 24 tells us that  $\varphi(P) \in \mathcal{F}(\varphi)$ . This completes the proof that  $\varphi(\mathcal{F}(\varphi)) \subset \mathcal{F}(\varphi)$ , which is the other inclusion.  $\square$



## 9. GOOD REDUCTION AND THE FATOU SET

Roughly speaking, a morphism  $\varphi : \mathbb{P}_K^N \rightarrow \mathbb{P}_K^N$  has good reduction at a point  $P \in \mathbb{P}^N(K)$  if the reduction of  $\varphi$  to the residue field  $k$  of  $K$  is well behaved at the reduction of  $P$ . In this section we show that  $\varphi$  has good reduction at  $P$  if and only if  $\varphi$  is nonexpanding in a neighborhood of  $P$  (whose radius we specify exactly). We then show how the locus of good reduction for  $\varphi$  can be used to describe a subset of the Fatou set  $\mathcal{F}(\varphi)$ . We begin with some definitions.

**Definition.** The morphism  $\varphi$  has *good reduction* at  $P$  if there is a lift  $\Phi$  of  $\varphi$  and a lift  $x$  of  $P$  satisfying

$$\|x\| = 1 \quad \text{and} \quad \|\Phi\| = 1 \quad \text{and} \quad \|\Phi(x)\| = 1. \quad (25)$$

We write

$$U^{\text{good}}(\varphi) = \{P \in \mathbb{P}^N(K) : \varphi \text{ has good reduction at } P\}$$

for the set of points at which  $\varphi$  has good reduction, and we write  $U^{\text{bad}}(\varphi)$  for the complementary set where  $\varphi$  has bad reduction.

We say that  $\varphi$  has *orbital good reduction* at  $P$  if  $\mathcal{O}_\varphi(P) \subset U^{\text{good}}(\varphi)$ , i.e., if  $\varphi$  has good reduction at every point in the forward orbit of  $P$ . We denote the set of such points by

$$U^{\text{orb-gd}}(\varphi) = \{P \in \mathbb{P}^N(K) : \varphi \text{ has orbital good reduction at } P\}.$$

Equivalently,

$$U^{\text{orb-gd}}(\varphi) = \bigcap_{n=0}^{\infty} \varphi^{-n}(U^{\text{good}}(\varphi)).$$

*Remark 27.* Since any two lifts of  $\varphi$  differ by a constant, it is easy to see that  $\varphi$  has good reduction at  $P$  if and only if every minimal lift  $\Phi$  of  $\varphi$  and every lift  $x$  of  $P$  satisfying  $\|x\| = 1$  also satisfies  $\|\Phi(x)\| = 1$ .

*Remark 28.* It follows easily from Proposition 9 that if  $\varphi$  has (global) good reduction in the sense of Remark 12, then  $U^{\text{good}}(\varphi) = \mathbb{P}^N(K)$ . Conversely, if  $U^{\text{good}}(\varphi) = \mathbb{P}^N(K)$ , then  $\|\Phi(x)\| = \|x\|^d$  for all  $x \in (K^{N+1})^*$ , i.e.,  $g_\Phi$  is identically 0. Proposition 15 then implies that  $\hat{g}_\Phi$  is identically 0, and hence [15, Proposition 12] tells us that  $\varphi$  has (global) good reduction. In conclusion,  $|\text{Res}(\varphi)| = 1$  if and only if  $\varphi$  has good reduction at every point of  $\mathbb{P}^N(K)$ .

*Remark 29.* An alternative way to define  $\varphi$  having good reduction at  $P$  is the existence of a lift  $\Phi$  with  $K$ -integral coefficients and an  $x$  with  $K$ -integral coordinates so that the image point  $\Phi(x)$  has at least one coordinate that is a  $K$ -unit. This allows us to reduce modulo the

maximal ideal to obtain points defined over the residue field  $k$  of  $K$ , and we obtain the formulas

$$\tilde{\Phi}(\tilde{x}) = \widetilde{\Phi(x)} \in \mathbb{A}^{N+1}(k) \setminus \{0\} \quad \text{and} \quad \tilde{\varphi}(\tilde{P}) = \widetilde{\varphi(P)} \in \mathbb{P}^N(k).$$

*Remark 30.* Our ad hoc definition of good reduction is convenient for calculations, but we note that it is equivalent to the usual scheme theoretic definition. Thus let  $R$  be the ring of integers of  $K$  and let  $k$  be the residue field. A point  $P \in \mathbb{P}^N(K)$  induces a section  $s_P : \text{Spec}(R) \rightarrow \mathbb{P}_R^N$ , and we write  $\tilde{P} = s_P(\text{Spec}(k))$  for the intersection of the section with the special fiber  $\mathbb{P}_k^N \subset \mathbb{P}_R^N$ . Then  $\varphi$  has good reduction at  $P$  if there is a rational map  $\tilde{\varphi} : \mathbb{P}_R^N \rightarrow \mathbb{P}_R^N$  whose restriction to the generic fiber  $\mathbb{P}_K^N$  is  $\varphi$  and such that  $\tilde{\varphi}$  is defined at  $\tilde{P}$ .

**Proposition 31.** *Let  $\varphi : \mathbb{P}_K^N \rightarrow \mathbb{P}_K^N$  be a morphism of degree  $d \geq 2$ , let  $\Phi$  be a minimal lift of  $\varphi$ , and let  $P \in \mathbb{P}^N(K)$ . Consider the following five statements.*

- (a)  $P \in U^{\text{good}}(\varphi)$ .
- (b)  $D_{|\text{Res}(\varphi)|}(P) \subset U^{\text{good}}(\varphi)$ .
- (c)  $g_\Phi(P) = 0$ .
- (d)  $\Delta(\varphi(Q), \varphi(R)) \leq \Delta(Q, R)$  for all  $Q, R \in D_{|\text{Res}(\varphi)|}(P)$ .
- (e)  $\varphi$  is nonexpanding in some neighborhood of  $P$ .

*Then we have the following implications:*

$$(a) \iff (b) \iff (c) \implies (d) \implies (e) \tag{26}$$

*In particular,  $U^{\text{good}}(\varphi)$  is an open set.*

*Proof.* It is clear that (b) implies (a) and (d) implies (e). For the remainder of this proof we fix a lift  $x$  of  $P$  satisfying  $\|x\| = 1$ .

We first prove that (a) implies (b), so let  $P \in U^{\text{good}}(\varphi)$ . The good reduction condition (25) tells us that  $\|\Phi(x)\| = 1$ . Now let  $Q \in D_{|\text{Res}(\varphi)|}(P)$  and choose a lift  $y$  of  $Q$  satisfying  $\|y\| = 1$ . Then Theorem 13 tells us that  $\|\Phi(x)\| = \|\Phi(y)\|$ , so  $y$  also satisfies the good reduction conditions (25). Hence  $Q \in U^{\text{good}}(\varphi)$ .

We next prove that (a) implies (d), so let  $P \in U^{\text{good}}(\varphi)$  and let  $Q, R \in D_{|\text{Res}(\varphi)|}(P)$ . In particular, since we already proved that (a) implies (b), we see that  $Q, R \in U^{\text{good}}(\varphi)$ . Hence if we choose lifts  $y$  of  $Q$  and  $z$  of  $R$  satisfying  $\|y\| = \|z\| = 1$ , then the definition of good reduction implies that

$$\|\Phi(y)\| = \|\Phi(z)\| = 1.$$

Writing  $\Phi = (\Phi_0, \dots, \Phi_N)$ , we proved earlier (see (5) in the proof of Theorem 10) that

$$|\Phi_i(y)\Phi_j(z) - \Phi_j(y)\Phi_i(z)| \leq \Delta(Q, R). \quad (27)$$

Dividing by  $\|\Phi(y)\| = \|\Phi(z)\| = 1$  yields  $\Delta(\varphi(Q), \varphi(R)) \leq \Delta(Q, R)$ .

It remains to show that (a) and (c) are equivalent. By definition, the function  $g_\Phi$  is given by

$$g_\Phi(P) = \frac{1}{d} \log \|\Phi(x)\| - \log \|x\|.$$

We have normalized  $x$  to satisfy  $\|x\| = 1$  and by definition,  $\varphi$  has good reduction at  $P$  if and only if  $\|\Phi(x)\| = 1$ . Hence  $P \in U^{\text{good}}(\varphi)$  if and only if  $g_\Phi(P) = 0$ .

This completes the proof of the implications (26). Finally, it is clear from (a)  $\Rightarrow$  (b) that  $U^{\text{good}}(\varphi)$  is an open set.  $\square$

We conclude with a proposition describing the set of points of orbital good reduction.

**Proposition 32.** *Let  $\varphi : \mathbb{P}_K^N \rightarrow \mathbb{P}_K^N$  be a morphism of degree  $d \geq 2$  and let  $\Phi$  be a minimal lift of  $\varphi$ .*

(a) *Let  $P \in U^{\text{orb-gd}}(\varphi)$  and let  $Q, R \in D_{|\text{Res}(\varphi)|}(P)$ . Then*

$$\Delta(\varphi^n(Q), \varphi^n(R)) \leq \Delta(Q, R) \quad \text{for all } n \geq 1. \quad (28)$$

(b) *Let  $P \in U^{\text{orb-gd}}(\varphi)$ . Then*

$$D_{|\text{Res}(\varphi)|}(P) \subset U^{\text{orb-gd}}(\varphi). \quad (29)$$

*In particular,  $U^{\text{orb-gd}}(\varphi)$  is an open set.*

(c)  $U^{\text{orb-gd}}(\varphi) = \{P \in \mathbb{P}^N(K) : \hat{g}_\Phi(P) = 0\} \subseteq \mathcal{F}(\varphi). \quad (30)$

(d)  $\hat{g}_\Phi$  is strictly negative on  $U^{\text{bad}}(\varphi)$ .

(e) The set  $\{P \in \mathbb{P}^N(K) : \hat{g}_\Phi(P) = 0\}$  is an open set.

*Proof.* (a,b) Let  $P \in U^{\text{good}}(\varphi)$ . We first use induction on  $n$  to prove (28) with  $R = P$ . It is clearly true for  $n = 1$ . Let  $Q \in D_{|\text{Res}(\varphi)|}(P)$  and assume that (28) with  $R = P$  is true for  $n$ . Then in particular we have

$$\Delta(\varphi^n(Q), \varphi^n(P)) \leq \Delta(Q, P) \leq |\text{Res}(\varphi)|.$$

Thus  $\varphi^n(Q) \in D_{|\text{Res}(\varphi)|}(\varphi^n(P))$ , and we know that  $\varphi^n(P) \in U^{\text{good}}(\varphi)$ , so Proposition 31(d) tells us that

$$\Delta(\varphi^{n+1}(Q), \varphi^{n+1}(P)) \leq \Delta(\varphi^n(Q), \varphi^n(P)).$$

Then the induction hypothesis gives  $\Delta(\varphi^{n+1}(Q), \varphi^{n+1}(P)) \leq \Delta(Q, P)$ . This proves that (28) is true for  $R = P$ .

In particular, we have shown that if  $Q \in D_{|\text{Res}(\varphi)|}(P)$ , then

$$\varphi^n(Q) \in D_{|\text{Res}(\varphi)|}(\varphi^n(P)) \quad \text{for all } n \geq 0.$$

By assumption we have  $\varphi^n(P) \in U^{\text{good}}$ , so Proposition 31(b) implies that  $\varphi^n(Q) \in U^{\text{good}}(\varphi)$ . This holds for all  $n \geq 0$ , hence  $Q \in U^{\text{orb-gd}}(\varphi)$ , which proves the inclusion (29). And clearly (29) implies that  $U^{\text{orb-gd}}(\varphi)$  is an open set.

We now show that (a) is true for all  $Q, R \in D_{|\text{Res}(\varphi)|}(P)$ . From (b) we have  $Q \in U^{\text{orb-gd}}(\varphi)$ . Further,

$$\Delta(R, Q) \leq \max\{\Delta(R, P), \Delta(Q, P)\} \leq |\text{Res}(\varphi)|,$$

so  $R \in D_{|\text{Res}(\varphi)|}(Q)$ . Hence we can apply our preliminary version of (a) to the point  $Q \in U^{\text{orb-gd}}(\varphi)$  and the point  $R \in D_{|\text{Res}(\varphi)|}(Q)$  to deduce that

$$\Delta(\varphi^n(Q), \varphi^n(R)) \leq \Delta(Q, R) \quad \text{for all } n \geq 1.$$

(c) We have

$$\begin{aligned} \hat{g}_\Phi(P) = 0 &\iff g_\Phi(\varphi^n(P)) = 0 \quad \text{for all } n \geq 0 \text{ (Proposition 15),} \\ &\iff \varphi^n(P) \in U^{\text{good}}(\varphi) \quad \text{for all } n \geq 0 \text{ (Theorem 31),} \\ &\iff P \in U^{\text{orb-gd}}(\varphi). \end{aligned}$$

This proves the lefthand equality in (30)

Next let  $P \in U^{\text{orb-gd}}(\varphi)$ . Then (a) says that the iterates of  $\varphi$  are nonexpanding on the disk  $D_{|\text{Res}(\varphi)|}(P)$ . This is much stronger than the assertion that  $\varphi$  is equicontinuous at every point in the disk. Hence  $P \in \mathcal{F}(\varphi)$ . This completes the proof that  $U^{\text{orb-gd}}(\varphi) \subset \mathcal{F}(\varphi)$ .

(d) From (c) we see that

$$P \in U^{\text{bad}}(\varphi) \iff P \notin U^{\text{good}}(\varphi) \implies P \notin U^{\text{orb-gd}}(\varphi) \implies \hat{g}_\Phi(P) \neq 0.$$

However, Proposition 15 tells us that  $\hat{g}_\Phi$  is nonpositive, so  $\hat{g}_\Phi(P) \neq 0$  is equivalent to  $\hat{g}_\Phi(P) < 0$ .

(e) This is immediate from (b) and (c).  $\square$

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